Graph sketching

1 Sketch the graph of the function $y = f(x) = x \cdot e^{-x}$.

a. State the domain restrictions and find the vertical asymptotes, if any.
b. Determine whether $f(x)$ is symmetric about the origin or is symmetric about the y-axis.
c. Find the intercepts with the coordinate axes.
d. Determine the sign of $f(x)$.
e. Determine the end behaviour of the graph.
f. Calculate the first derivative and determine the critical points.
g. Analyze the sign of $f'(x)$.
h. Calculate the second derivative.
i. Analyze the sign of $f''(x)$ and find the points of inflection, if any.
j. Sketch the graph.

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a. The ______ is $(-\infty\,;\,+\infty)$; therefore there are no vertical ________________

b. $f(-x) = -x \cdot e^{x}$ and $-f(x) = -x \cdot e^{-x}$.
______ $f(x) \neq f(-x)$ and $f(x) \neq -f(x)$ there are no symmetries.

c. _______ $x = 0$ into $y = x \cdot e^{-x} \longrightarrow y = 0$.
Therefore $f(x)$ _____ through the origin.

d. Consider $x \cdot e^{-x} > 0$
First solve the related equation $x \cdot e^{-x} = 0$.
Apply the zero product property

$x = 0$

The equation $e^{-x} = 0$ has no solution.

Now determine the sign of $f(x)$.

		0	
$x > 0$	− − − − −	0	+ + + + +
$e^{-x} > 0$	+ + + + + +		+ + + + + +
f(x)	− − − − −	0	+ + + + +

From the sign chart we see that $f(x) < 0$ on interval $(-\infty\,;\,0)$ and $f(x) > 0$ ___ interval $(0\,;\,+\infty)$.

e. $\lim_{x \to -\infty} x \cdot e^{-x} = -\infty$

$\lim_{x \to +\infty} x \cdot e^{-x} = \lim_{x \to +\infty} \frac{x}{e^{x}} = \frac{\infty}{\infty}$

Apply De l'Hôpital's theorem

$\lim_{x \to +\infty} \frac{x}{e^{x}} = \lim_{x \to +\infty} \frac{1}{e^{x}} = 0$

This limit shows that the graph get _____ and closer to the x-axis as x ______ without bound.

f. $y' = e^{-x} + (-e^{-x})x \longrightarrow y' = e^{-x}(1 - x).$

Find the critical points. Let $y' = 0$

$e^{-x}(1 - x) = 0$

Apply the zero product property.

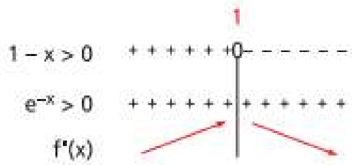
The equation $e^{-x} = 0$ has no solution.

The equation $1 - x = 0$ is true for $x = 1$.

Substitute $x = 1$ into $y = x \cdot e^{-x} \longrightarrow y = e^{-1} \longrightarrow y = \frac{1}{e}$

$M\left(1\,;\,\frac{1}{e}\right)$ is a critical point of $f(x)$.

g. Use a sign chart to determine the sign of $f'(x)$.



The sign chart shows that

$f'(x) > 0$ on interval $(-\infty\,;\,1)$, hence $f(x)$ is increasing in this interval;

$f'(x) < 0$ on interval $(1\,;\,+\infty)$, hence $f(x)$ is ________ in this interval.

We conclude that $f(x)$ has a maximum at $x = 1$.

f. $y' = e^{-x} + (-e^{-x})x \longrightarrow y' = e^{-x}(1 - x).$

Find the critical points. Let $y' = 0$

$e^{-x}(1 - x) = 0$

Apply the zero product property.

The equation $e^{-x} = 0$ has no solution.

The equation $1 - x = 0$ is true for $x = 1$.

Substitute $x = 1$ into $y = x \cdot e^{-x} \longrightarrow y = e^{-1} \longrightarrow y = \frac{1}{e}$

$M\left(1\,;\,\frac{1}{e}\right)$ is a critical point of $f(x)$.

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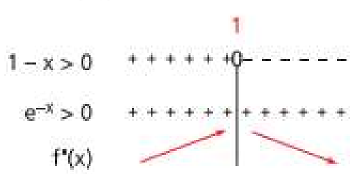


The sign chart shows that

$f'(x) > 0$ on interval $(-\infty\,;\,1)$, hence $f(x)$ is increasing in this interval;

$f'(x) < 0$ on interval $(1\,;\,+\infty)$, hence $f(x)$ is ________ in this interval.

We conclude that $f(x)$ has a ________ at $x = 1$.

h. $y'' = -e^{-x}(1-x) + (-1)e^{-x} \longrightarrow y'' = -e^{-x}(1-x+1) \longrightarrow y'' = e^{-x}(x-2)$

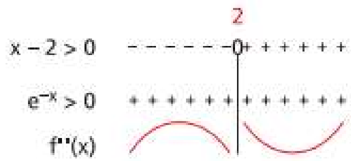
i. Consider $e^{-x}(x-2) > 0$. First solve the related ______ $e^{-x}(x-2) = 0$.

Apply the zero product ____________

The equation $e^{-x} = 0$ has no ________

The equation $x - 2 = 0$ is true for $x = 2$.

Now determine the sign of $f''(x)$.



We conclude that $f(x)$ is concave down on interval $(-\infty\,;\,2)$ and is concave up on interval $(2\,;\,+\infty)$. Hence $f(x)$ has a point of inflection at $x = 2$.

Substitute $x = 2$ into $y = x \cdot e^{-x} \longrightarrow y = 2 \cdot e^{-2} \longrightarrow y = \dfrac{2}{e^2}$

The point $F\left(2\,;\,\dfrac{2}{e^2}\right)$ is an ______ point of $f(x)$.

j. Finally sketch the graph (**FIGURA 1**).

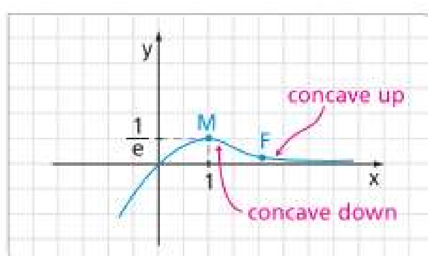


FIGURE 1