## Graph sketching

**1** Sketch the graph of the function $y = f(x) = x \cdot e^{-x}$.

**a.** State the domain restrictions and find the vertical asymptotes, if any.

**b.** Determine whether $f(x)$ is symmetric about the origin or is symmetric about the $y$-axis.

**c.** Find the intercepts with the coordinate axes.

**d.** Determine the sign of $f(x)$.

**e.** Determine the end behaviour of the graph.

**f.** Calculate the first derivative and determine the critical points.

**g.** Analyze the sign of $f'(x)$.

**h.** Calculate the second derivative.

**i.** Analyze the sign of $f''(x)$ and find the points of inflection, if any.

**j.** Sketch the graph.

**a.** The domain is $(-\infty\,;\,+\infty)$; therefore there are no vertical asymptotes.

**b.** $f(-x) = -x \cdot e^{x}$ and $-f(x) = -x \cdot e^{-x}$.
Since $f(x) \neq f(-x)$ and $f(x) \neq -f(x)$ there are no symmetries.

**c.** Substitute $x = 0$ into $y = x \cdot e^{-x} \longrightarrow y = 0$.
Therefore $f(x)$ passes through the origin.

**d.** Consider $x \cdot e^{-x} > 0$
First solve the related equation $x \cdot e^{-x} = 0$.
Apply the zero product property

$x = 0$

The equation $e^{-x} = 0$ has no solution.

Now determine the sign of $f(x)$.

| | | 0 | |
|---|---|---|---|
| $x > 0$ | − − − − − − | 0 | + + + + + + |
| $e^{-x} > 0$ | + + + + + + | | + + + + + + |
| $f(x)$ | − − − − − − | 0 | + + + + + + |

From the sign chart we see that $f(x) < 0$ on interval $(-\infty\,;\,0)$ and $f(x) > 0$ on interval $(0\,;\,+\infty)$.

**e.** $\lim\limits_{x \to -\infty} x \cdot e^{-x} = -\infty$

$$\lim_{x \to +\infty} x \cdot e^{-x} = \lim_{x \to +\infty} \frac{x}{e^{x}} = \frac{\infty}{\infty}$$

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Apply De l'Hôpital's theorem

$$\lim_{x \to +\infty} \frac{x}{e^x} = \lim_{x \to +\infty} \frac{1}{e^x} = 0$$

This limit shows that the graph get closer and closer to the $x$-axis as $x$ increase without bound.

**f.** $y' = e^{-x} + (-e^{-x})x \longrightarrow y' = e^{-x}(1 - x).$

Find the critical points. Let $y' = 0$

$e^{-x}(1 - x) = 0$

Apply the zero product property.

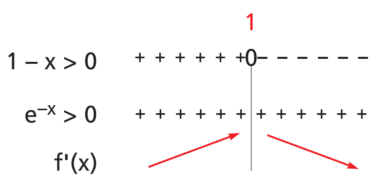
The equation $e^{-x} = 0$ has no solution.

The equation $1 - x = 0$ is true for $x = 1$.

Substitute $x = 1$ into $y = x \cdot e^{-x} \longrightarrow y = e^{-1} \longrightarrow y = \dfrac{1}{e}$

$M\left(1\ ;\ \dfrac{1}{e}\right)$ is a critical point of $f(x)$.

**g.** Use a sign chart to determine the sign of $f'(x)$.




The sign chart shows that

$f'(x) > 0$ on interval $(-\infty\ ;\ 1)$, hence $f(x)$ is increasing in this interval;

$f'(x) < 0$ on interval $(1\ ;\ +\infty)$, hence $f(x)$ is decreasing in this interval.

We conclude that $f(x)$ has a maximum at $x = 1$.

**h.** $y'' = -e^{-x}(1 - x) + (-1)e^{-x} \longrightarrow y'' = -e^{-x}(1 - x + 1) \longrightarrow y'' = e^{-x}(x - 2)$

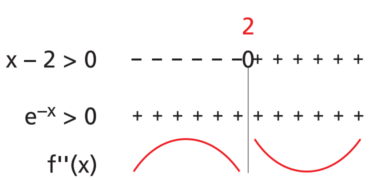
**i.** Consider $e^{-x}(x - 2) > 0$. First solve the related equation $e^{-x}(x - 2) = 0$.

Apply the zero product property.

The equation $e^{-x} = 0$ has no solution.

The equation $x - 2 = 0$ is true for $x = 2$.

Now determine the sign of $f''(x)$.




We conclude that $f(x)$ is concave down on interval $(-\infty\ ;\ 2)$ and is concave up on interval $(2\ ;\ +\infty)$. Hence $f(x)$ has a point of inflection at $x = 2$.

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Substitute $x = 2$ into $y = x \cdot e^{-x} \longrightarrow y = 2 \cdot e^{-2} \longrightarrow y = \frac{2}{e^2}$

The point $F\left(2\,;\,\frac{2}{e^2}\right)$ is an inflection point of $f(x)$.

**j.** Finally sketch the graph (**FIGURA 1**).

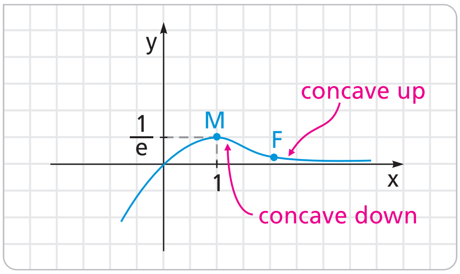


**FIGURE 1**

## What does it mean?

**First derivative, second derivative** Derivata prima, derivata seconda

**Point of inflection** Punto di flesso

**Zero product property** Legge di annullamento del prodotto



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